THIRD-ORDER EFFECTS IN THE PROPAGATION OF ELASTIC WAVES IN ISOTROPIC SOLIDS

GENERATION OF HIGHER HARMONICS

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The nonlinear interactions of longitudinal and shear waves in an isotropic solid medium are discussed in the nine-constant theory of elasticity. Expressions are obtained describing the generation of second and third harmonics of elastic waves in the approximations of a nondispersive medium and in the presence of quasistatic elastic fields.

1. The theory of nonlinear elastic-wave interactions in solids is limited mainly to the nonlinear five-constant theory of elasticity [1]. Experiments show [1-4], that in the propagation of elastic waves in an isotropic medium there occur effects associated with cubic nonlinearity of the solid medium such as the generation of second shear-wave harmonics, and in the presence of static stresses, of third longitudinal and shear-wave harmonics, etc. A full theoretical treatment of these effects for elastic waves has not been undertaken.

In the present paper we consider the interaction of unmodulated elastic waves on the basis of aysmptotic methods [5] in the nonlinear, nine-constant theory of elasticity.

In order to treat these questions we solve to third order the equations of motion, which in Lagrangian coordinates take the form [1]

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial}{\partial x_j}\left[\frac{\partial \mathscr{E}}{\partial (\partial u_i / \partial x_j)}\right] \tag{1.1}$$

where ρ_0 is the density of the unperturbed medium, u_i are the components of the displacement vector, t is the time, x_j are the coordinates, and ε is the internal energy per unit volume. In order to obtain the equations of motion with allowance for a cubic elastic nonlinearity, we put ε in the form

$$\mathscr{E} = \mu u_{ik}^2 + (K/2 - \mu/3)\, u_{ll}^2 + {}^1/_3\, A u_{il} u_{ik} u_{lk} + B u_{ik}^2 u_{ll} + {}^1/_3 C u_{ll}^3 + D u_{ll}^4 + G u_{ik} u_{il} u_{kl} u_{jj}$$
$$+ H u_{ik}^2 u_{ll}^2 + J u_{ik}^4 \quad u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right) \tag{1.2}$$

where u_{ik} is the strain tensor, μ and K are the second-order elastic moduli, A, B, C are the third-order elastic moduli, and D, G, H, and J are the fourth-order elastic moduli. The fourth-order elastic moduli can also be introduced in terms of the expansion of ε in invariants of the strain tensor [6].

$$\mathscr{E}_4 = n_1 I_1^4 + n_2 I_1^2 I_2 + n_3 I_1 I_3 + n_4 I^2,$$
$$I_1 = u_{ll}, \quad I_2 = {}^1/_2\,(u_{ll}^2 - u_{ik}^2), \quad I_3 = \det(u_{ik}) \tag{1.3}$$

where ε_4 is the part of ε containing terms of fourth degree in the strains. The connection between the fourth-order moduli in (1.2) and (1.3) is given by the relations

$$D = n_1 + n_2/2 + n_3/6 + n_4/4, \quad H = -{}^1/_2\,(n_2 + n_3 + n_4)$$
$$G = n_3/3, \quad J = n_4/4 \tag{1.4}$$

After substitution (1.2) into (1.1) we obtain the equations of motion describing the nonlinear interactions of elastic waves with allowance for cubic nonlinearity of the solid medium:

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$$\rho_0 \partial^2 u_i / \partial t^2 - \mu \partial^2 u_i / \partial x^2{}_k - (K + \mu / 3)\, \partial^2 u_l / \partial x_l\, \partial x_i = f_i' + f_i'' \tag{1.5}$$

where f_i' is determined by the quadratic nonlinearity of the solid medium

$$f_i' = \left(\mu + \frac{A}{4}\right)\left(\frac{\partial^2 u_l}{\partial x_k^2}\frac{\partial u_l}{\partial x_i} + \frac{\partial u_i}{\partial x_l}\frac{\partial^2 u_l}{\partial x_k^2} + 2\frac{\partial u_l}{\partial x_k}\frac{\partial^2 u_i}{\partial x_l \partial x_k}\right) + \left(K + \frac{\mu}{3} + \frac{A}{4} + B\right)\times$$
$$\times\left(\frac{\partial u_l}{\partial x_k}\frac{\partial^2 u_l}{\partial x_i \partial x_k} + \frac{\partial u_i}{\partial x_l}\frac{\partial^2 u_k}{\partial x_l \partial x_k}\right) + \left(K - \frac{2\mu}{3} + B\right)\frac{\partial u_l}{\partial x_l}\frac{\partial^2 u_i}{\partial x_k^2} + \left(\frac{A}{4} + B\right)\left(\frac{\partial^2 u_k}{\partial x_l \partial x_k}\frac{\partial u_l}{\partial x_i} + \frac{\partial u_k}{\partial x_l}\frac{\partial^2 u_l}{\partial x_i \partial x_k}\right) + (B + 2C)\frac{\partial u_l}{\partial x_l}\frac{\partial^2 u_k}{\partial x_k \partial x_i}$$

and f_i'' is determined by the cubic elastic nonlinearity of the solid medium

$$f_i'' = (12D + H)\left(\frac{\partial u_l}{\partial x_l}\right)^2 \frac{\partial^2 u_k}{\partial x_k \partial x_i} + \left(J + B + \frac{K}{2} - \frac{\mu}{3}\right)\left(\frac{\partial u_l}{\partial u_m}\right)^2 \frac{\partial^2 u_i}{\partial x_k^2} + \left(J + \frac{B}{2}\right)\times$$
$$\times\frac{\partial u_k}{\partial x_m}\frac{\partial u_m}{\partial x_k}\frac{\partial^2 u_i}{\partial x_l^2} + (J + H)\frac{\partial u_l}{\partial x_m}\frac{\partial u_m}{\partial x_l}\frac{\partial^2 u_k}{\partial x_k \partial x_i} + \left(J + H + C + \frac{B}{2}\right)\left(\frac{\partial u_l}{\partial x_m}\right)^2 \frac{\partial^2 u_k}{\partial x_k \partial x_i} +$$
$$+ \left(2J + 2B + \frac{A}{2} + K + \frac{\mu}{3}\right)\frac{\partial u_i}{\partial x_k}\frac{\partial u_l}{\partial x_m}\frac{\partial^2 u_l}{\partial x_m \partial x_k} + \left(2J + B + \frac{A}{4}\right)\times$$
$$\times\left(\frac{\partial u_k}{\partial x_i}\frac{\partial u_l}{\partial x_m}\frac{\delta^2 u_l}{\partial x_m \partial x_k} + \frac{\partial u_i}{\partial x_k}\frac{\partial u_m}{\partial x_l}\frac{\partial^2 u_l}{\partial x_m \partial x_k}\right) + 2J\frac{\partial u_k}{\partial x_i}\frac{\partial u_m}{\partial x_l}\frac{\partial^2 u_l}{\partial x_m \partial x_k} + \left(\frac{3G}{4} + B\right)\times$$
$$\times\left(\frac{\partial u_m}{\partial x_m}\frac{\partial u_i}{\partial x_l}\frac{\partial^2 u_l}{\partial x_k^2} + 2\frac{\partial u_m}{\partial x_m}\frac{\partial u_k}{\partial x_l}\frac{\partial^2 u_i}{\partial x_l \partial x_k} + \frac{\partial u_m}{\partial x_m}\frac{\partial u_l}{\partial x_i}\frac{\partial^2 u_l}{\partial x_k^2}\right) + \left(\frac{3G}{4} + \frac{A}{4}\right)\times$$
$$\times\left(\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}\frac{\partial^2 u_m}{\partial x_m \partial x_k} + \frac{\partial u_i}{\partial x_l}\frac{\partial u_l}{\partial x_k}\frac{\partial^2 u_m}{\partial x_m \partial x_k}\right) + \frac{3G}{4}\left(\frac{\partial u_l}{\partial x_i}\frac{\partial u_k}{\partial x_l}\frac{\partial^2 u_m}{\partial x_m \partial x_k} +\right.$$
$$\left.+ \frac{\partial u_m}{\partial x_k}\frac{\partial u_l}{\partial x_m}\frac{\partial^2 u_k}{\partial x_l \partial x_i}\right) + \left(\frac{3G}{4} + B + \frac{A}{4}\right)\left(\frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\frac{\partial^2 u_m}{\partial x_m \partial x_k} + \frac{\partial u_m}{\partial x_k}\frac{\partial u_m}{\partial x_l}\frac{\partial^2 u_k}{\partial x_l \partial x_i} +\right.$$
$$\left.+ \frac{\partial u_k}{\partial x_m}\frac{\partial u_m}{\partial x_l}\frac{\partial^2 u_k}{\partial x_l \partial x_i} + \frac{\partial u_k}{\partial x_m}\frac{\partial u_l}{\partial x_m}\frac{\partial^2 u_l}{\partial x_k \partial x_i}\right) + \left(\frac{3G}{4} + 2H\right)\left(\frac{\partial u_m}{\partial x_m}\frac{\partial u_k}{\partial x_l}\frac{\partial^2 u_l}{\partial x_k \partial x_i} +\right.$$
$$\left.+ \frac{\partial u_m}{\partial x_m}\frac{\partial u_l}{\partial x_i}\frac{\partial^2 u_k}{\partial x_k \partial x_l}\right) + \left(\frac{3G}{4} + 2H + 2C + B\right)\left(\frac{\partial u_m}{\partial x_m}\frac{\partial u_i}{\partial x_l}\frac{\partial^2 u_k}{\partial x_k \partial x_l} +\right.$$
$$\left.+ \frac{\partial u_m}{\partial x_m}\frac{\partial u_l}{\partial x_k}\frac{\partial^2 u_l}{\partial x_k \partial x_i}\right) + (C + H)\left(\frac{\partial u_l}{\partial x_l}\right)^2 \frac{\partial^2 u_i}{\partial x_k^2} + \left(\frac{A}{2} + \mu\right)\left(\frac{\partial u_i}{\partial x_l}\frac{\partial u_m}{\partial x_l}\frac{\partial^2 u_m}{\partial x_k^2} +\right.$$
$$\left.+ \frac{\partial u_m}{\partial x_k}\frac{\partial u_m}{\partial x_l}\frac{\partial^2 u_i}{\partial x_l \partial x_k}\right) + \frac{A}{4}\left(\frac{\partial u_l}{\partial x_i}\frac{\partial u_m}{\partial x_l}\frac{\partial^2 u_m}{\partial x_k^2} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_l}\frac{\partial^2 u_l}{\partial x_k^2} + \frac{\partial u_k}{\partial x_l}\frac{\partial u_m}{\partial x_l}\frac{\partial^2 u_i}{\partial x_k \partial x_l} +\right.$$
$$\left.+ \frac{\partial u_k}{\partial x_l}\frac{\partial u_i}{\partial x_k}\frac{\partial^2 u_l}{\partial x_m^2}\right) + \frac{A}{2}\left(\frac{\partial u_k}{\partial x_l}\frac{\partial u_m}{\partial x_i}\frac{\partial^2 u_m}{\partial x_k \partial x_l} + \frac{\partial u_l}{\partial x_k}\frac{\partial u_m}{\partial x_l}\frac{\partial^2 u_i}{\partial x_k \partial x_m} + \frac{\partial u_l}{\partial x_m}\frac{\partial u_i}{\partial x_k}\frac{\partial^2 u_k}{\partial x_l \partial x_m}\right) +$$
$$+ B\left(\frac{\partial u_l}{\partial x_k}\frac{\partial u_i}{\partial x_k}\frac{\partial^2 u_m}{\partial x_m \partial x_l} + \frac{\partial u_l}{\partial x_k}\frac{\partial u_i}{\partial x_l}\frac{\partial^2 u_m}{\partial x_m \partial x_k}\right)$$

The contribution from terms of fourth order in powers of $\partial u_i/\partial x_k$ in (1.1) to the four-wave interactions of plane elastic waves considered below is $(\mu^*)^{-1}$ times less than the contribution from the third-order terms, where $\mu^* = U/\lambda$ is the Mach number. Actual experimentally attainable values of μ^* are 10^{-5}–10^{-6}, and hence terms of fourth and higher order in powers of $\partial u_i/\partial x_k$ in (1.5) are not considered.

Equation (1.5) is the general equation of motion for a solid, nondissipative medium in the cubic approximation. When analyzing its solutions, we restrict ourselves to plane elastic waves, for which (1.5) splits up into three equations. Let the plane longitudinal and shear waves propagate along $x_1 = x$. In this case for a longitudinal wave we obtain an equation of the form

$$\rho_0 \frac{\partial^2 u_x}{\partial t^2} - \rho_0 c_l^2 \frac{\partial^2 u_x}{\partial x^2} = \beta \frac{\partial u_x}{\partial x}\frac{\partial^2 u_x}{\partial x^2} + \gamma\left(\frac{\partial u_y}{\partial x}\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial u_z}{\partial x}\frac{\partial^2 u_z}{\partial x^2}\right) +$$
$$+ \tau_1 \left(\frac{\partial u_x}{\partial x}\right)^2 \frac{\partial^2 u_x}{\partial x^2} + \tau_2 \frac{\partial u_x}{\partial x}\left(\frac{\partial u_y}{\partial x}\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial u_z}{\partial x}\frac{\partial^2 u_z}{\partial x^2}\right) + \tau_3\left[\left(\frac{\partial u_y}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial x}\right)^2\right]\frac{\partial^2 u_x}{\partial x^2} \tag{1.6}$$
$$\rho_0 c_l^2 = K + {}^4/_3\mu, \quad \beta = 3K + 4\mu + 2A + 6B + 2C, \quad \gamma = \mu + {}^1/_2 A + B$$
$$\tau_1 = {}^3/_2 K + 2\mu + 18B + 6A + 6C + 12\,(D + G + H + J)$$
$$\tau_2 = K + {}^4/_3\mu + {}^7/_2 B + {}^5/_2 A + 2C + 4J + 3G + 2H, \quad \tau_3 = {}^1/_2 K + {}^2/_3\mu +$$
$$+ {}^7/_2 B + C + {}^5/_4 A + 2J + {}^3/_2 G + H$$

For a shear wave, polarized along the y axis, we have

$$\rho_0 \frac{\partial^2 u_y}{\partial t^2} - \rho_0 c_t^2 \frac{\partial^2 u_y}{\partial x^2} = \gamma\left(\frac{\partial u_x}{\partial x}\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial u_y}{\partial x}\frac{\partial^2 u_x}{\partial x^2}\right) +$$
$$+ \tau_3 \left(\frac{\partial u_x}{\partial x}\right)^2 \frac{\partial^2 u_y}{\partial x^2} + \tau_4 \left(\frac{\partial u_z}{\partial x}\right)^2 \frac{\partial^2 u_y}{\partial x^2} + 2\tau_4 \frac{\partial u_z}{\partial x}\frac{\partial u_y}{\partial x}\frac{\partial^2 u_z}{\partial x^2} \tag{1.7}$$
$$+ \tau_2 \frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial x}\frac{\partial^2 u_x}{\partial x^2} + \tau_5 \left(\frac{\partial y_y}{\partial x}\right)^2 \frac{\partial^2 u_y}{\partial x^2} \quad \rho_0 c_t^2 = \mu, \quad \tau_4 = {}^1/_2 K + {}^2/_3\mu + B + {}^1/_2 A + J, \quad \tau_5 = {}^3/_2 K + 2\mu + 3B + {}^3/_2 A + 3J$$

where γ, τ_2, and τ_3 are the same as in (1.6).

For shear waves, polarized along the z axis, the equation will be analogous to (1.7). To take into account the dissipative properties of the solid medium, it is necessary to introduce terms into the left sides of equations (1.6) and (1.7) of the form

$$-\left({}^4/_3\eta + \xi\right)\frac{\partial^3 u_x}{\partial t \partial x^2} \text{ and } -\eta\frac{\partial^3 u_y}{\partial t \partial x^2}$$

respectively, where η is the shear viscosity coefficient, and ξ is the bulk viscosity coefficient.

According to the general theory of nonlinear one-dimensional waves [5], media with a cubic nonlinearity are characterized by the capacity for four-wave interactions, i.e., they admit third-harmonic generation. In the presence of static shear fields in such a medium three-wave interactions are also possible, i.e., second-harmonic generation.

2. Let us consider the generation of higher harmonics of longitudinal and shear elastic waves. We shall look for a solution to equations (1.6) and (1.7) in the form

$$U = \sum_{n=1}^{3} \mathrm{Re}\left[A_n(\mu^* x, \mu^* t)\, e^{in(\omega t - kx)}\right] \qquad (2.1)$$

where $n\omega$, nk, and A_n are the frequency, wave vector, and complex amplitude of the n-th harmonic.

Employing the standard averaging procedure [5], we obtain the following reduced equations for a longitudinal elastic wave:

$$\begin{aligned}
&\frac{\partial A_1}{\partial t} + c_l\frac{\partial A_1}{\partial x} = -\alpha_1 c_l A_1 - \frac{k^2\beta}{\rho_0 c_l}\left(\bar{A}_1 A_2 + 3\bar{A}_2 A_3\right) + \\
&+ \frac{i\tau_1 k^3}{2\rho_0 c_l}\left(12 A_2^2 \bar{A}_3 - 3\bar{A}_1^2 A_3\right) + \frac{i\tau_1 k^3}{2\rho_0 c_l}\left(A_1\bar{A}_1 + 8A_2\bar{A}_2 + 18A_3\bar{A}_3\right) A_1 \\
&\frac{\partial A_2}{\partial t} + c_l\frac{\partial A_1}{\partial x} = -\alpha_2 c_l A_2 + \frac{k^2\beta}{4\rho_0 c_l}\left(A_1^2 - 6\bar{A}_1 A_3\right) \\
&+ \frac{6i\tau_1 k^3}{\rho_0 c_l} A_1\bar{A}_2 A_3 + \frac{i\tau_1 k^3}{\rho_0 c_l}\left(2A_1\bar{A}_1 + 4A_2\bar{A}_2 + 18A_3\bar{A}_3\right) A_2 \\
&\frac{\partial A_3}{\partial t} + c_l\frac{\partial A_3}{\partial x} = -\alpha_3 c_l A_3 + \frac{k^2\beta A_1 A_2}{\rho_0 c_l} - \frac{i\tau_1 k^3}{6\rho_0 c_l}\left(A_1^3 - 12\bar{A}_1 A_2^2\right) \\
&+ \left(6A_1\bar{A}_1 + 24A_2\bar{A}_2 + 27A_3\bar{A}_3\right) A_3 \quad \alpha_1 = \left({}^4/_3\eta + \xi\right)\omega^2 / 2c_l^3,\ \alpha_2 = 4\alpha_1,\ \alpha_3 = 9\alpha_1
\end{aligned} \qquad (2.2)$$

Here A_n is the complex conjugate of A_n.

The first terms on the right sides of the equations (2.2) describe the absorption of the elastic wave; the second terms describe the interaction of harmonics by means of the quadratic nonlinearity; the third terms describe the interaction of harmonics by means of the cubic nonlinearity; and the fourth terms describe the nonsynchronous interactions of the harmonics by means of the cubic nonlinearity (nonlinear detuning [5]).

In analyzing our system of reduced equations (2.2), we note that the process of generating the third harmonic of a longitudinal wave is connected both with the quadratic nonlinearity of the solid medium [the second terms in equations (2.2)] and with the cubic nonlinearity (third terms). In the five-constant theory of elasticity there is generated a second harmonic of a longitudinal elastic wave, growing in space [1, 7]. The third harmonic is generated as a result of two successive three-way interactions through the quadratic nonlinearity and one four-wave interaction through the cubic nonlinearity. The generation of harmonics has a cumulative nature in space.

In a dissipative, nondispersive medium a solution can be sought in the approximation of the given first-harmonic field [1, 5] i.e., under the conditions

$$\begin{aligned}
&A_1 = A_{10},\ A_2 = A_3 = 0 \quad \text{for} \quad x = 0 \\
&A_1(x) \gg A_2(x) \gg A_3(x) \quad \text{for} \quad x > 0
\end{aligned} \qquad (2.3)$$

The solution of system (2.2) taking (2.3) into account has the form

$$\begin{aligned}
&A_1 = A_{10} e^{-\alpha_1 x}, \quad A_2 = \frac{k^2\beta A_{10}^2}{4\rho_0 c_l^2(\alpha_2 - 2\alpha_1)}\left(e^{-2\alpha_1 x} - e^{-\alpha_2 x}\right) \equiv P(x) A_{10}^2 \\
&A_3 = \frac{k^4\beta^2 A_{10}^3}{4\rho_0^2 c_l^4(\alpha_2 - 2\alpha_1)}\left[\frac{e^{-3\alpha_1 x}}{\alpha_3 - 3\alpha_1} - \frac{e^{-(\alpha_1+\alpha_2)x}}{\alpha_3 - \alpha_2 - \alpha_1} + \frac{(\alpha_2 - 2\alpha_1)\, e^{-\alpha_3 x}}{(\alpha_3 - 3\alpha_1)(\alpha_3 - \alpha_2 - \alpha_1)}\right] \\
&- \frac{i\tau_1 k^3 A_{10}^3}{6\rho_0 c_l^2(\alpha_3 - 3\alpha_1)}\left(e^{-3\alpha_1 x} - e^{-\alpha_3 x}\right) \equiv \left[R_1(x) - iR_2(x)\right] A_{10}^3
\end{aligned} \qquad (2.4)$$

Going to real amplitudes and phases by means of the relations

$$A_{10} = a_{10} e^{i\varphi_{10}},\ A_2 = a_2 e^{i\varphi_2},\ A_3 = a_3 e^{i\varphi_3} \qquad (2.5)$$

we obtain the solution for the amplitude and phase of the second harmonic [7]

$$\varphi_2 - 2\varphi_{10} = 0, \quad a_2 = P(x)\, a_{10}^2 \tag{2.6}$$

The third harmonic for the cubic nonlinearity is generated with a phase shift of $\pi/2$ with respect to the case of two three-wave interactions, and for this harmonic we have the following expressions for the amplitude and phase, respectively:

$$\begin{aligned} a_3 &= a_{10}^3 \left[R_1^2(x) + R_2^2(x)\right]^{1/2} \\ \varphi_3 &= 3\varphi_{10} - \operatorname{arc\,tg}\left[R_2(x) / R_1(x)\right] \end{aligned} \tag{2.7}$$

Analyzing the solution (2.6), we note that the amplitude of the second and third harmonics will grow in proportion to x, reaching a maximum at some distance x = L. This distance is different for the second and third harmonics. In Fig. 1 we show graphically the dependences of the second-harmonic amplitude 10^8 a_2 cm (curve 1), the third-harmonic amplitude, generated with the quadratic nonlinearity, $10^9\, a_{10}^3 R_1(x)$ cm (curve 2), and the third-harmonic amplitude, generated with the cubic nonlinearity, $10^{11} R_2(x)\, a_{10}^3$cm (curve 3).

Here we chose actual mean values of the parameters of the materials (metals) ($\beta/\rho_0 c_l^2 = 20$, $\tau_1/\rho_0 c_l^2 = 500$, $\alpha = 10^{-3} 2\pi/\lambda$). When $\lambda = 10^{-1}$ cm and $c_l = 5 \cdot 10^5$ cm/sec and the first-harmonic amplitude is $a_{10} = 10^{-6}$ cm, the maximum value of the second harmonic is $a_2(L) = 3.9 \cdot 10^{-6}$ cm for L = 5.5 cm, and analogously for the components $a_{10}^3 R_1$ and $R_2\, a_{10}^3$ of the third harmonic we have the following maximum values: $a_{10}^3 R_1(L_1) = 3.7 \cdot 10^{-9}$ cm for $L_1 = 6.3$ cm and $a_{10}^3 R_2(L_2) = 2.15 \cdot 10^{-11}$ cm for $L_2 = 2.9$ cm. Thus, the contribution to the longitudinal-wave third-harmonic generation from the cubic nonlinearity is small compared to successive three-wave interactions in the quadratic term.

Turning to consideration of shear-wave third-harmonic generation, we note that in distinction to longitudinal waves in the five-constant elasticity theory there is no generation of the shear-wave second harmonic [1,7]. This means that generation of the third shear-wave harmonic will take place only due to the cubic elastic nonlinearity. The reduced equations for the complex shear-wave amplitudes, obtained after substitution (2.1) into (1.7), have the form

$$\begin{aligned} \frac{\partial A_1}{\partial t} + c_t \frac{\partial A_1}{\partial x} &= -\alpha_1 c_t A_1 - \frac{3ik^3\tau_5 \bar{A}_1^2 A_3}{2\rho_0 c_t} + \frac{ik^3\tau_5}{2\rho_0 c_t}\left(A_1\bar{A}_1 + 18A_3\bar{A}_3\right)A_1 \\ \frac{\partial A_3}{\partial t} + c_t \frac{\partial A_3}{\partial x} &= -\alpha_3 c_t A_3 - \frac{ik^3\tau_5 A_1^3}{6\rho_0 c_t} + \frac{ik^3\tau_5}{2\rho_0 c_t}\left(6A_1\bar{A}_1 + 27A_3\bar{A}_3\right)A_3 \\ \alpha_1 &= \eta\omega^2 / 2c_t^3, \quad \alpha_3 = 9\alpha_1 \end{aligned} \tag{2.8}$$

In the approximation of the given first-harmonic field we obtain the following solution to the system of equations (2.8):

$$\varphi_3 - 3\varphi_{10} + \frac{\pi}{2} = 0, \quad a_3 = \frac{\tau_5 k^3 a_{10}^3}{6\rho_0 c_t^2(\alpha_3 - 3\alpha_1)}\left(e^{-3\alpha_1 x} - e^{-\alpha_3 x}\right) \tag{2.9}$$

The graphical dependence of the amplitude of the third shear-wave harmonic is analogous to that given in Fig. 1 (curve 3).

3. The presence of static elastic fields in the solid medium leads to the generation of the second harmonic (three-wave interaction), the amplitude of which depends linearly on the magnitude of this field. This phenomenon can be viewed as a nonresonant parametric interaction in a medium with a cubic nonlinearity.

It is useful to look for a solution to the present problem in the form [8]

$$u = \sum_{n=1}^{2} \operatorname{Re}\left[A_n(\mu^* x, \mu^* t) e^{in(\omega t - kx)}\right] + u^m(x, y, z, t) \tag{3.1}$$

where A_n are the complex amplitudes of the corresponding harmonics, and u^m (x, y, z, t) is the internal or external (static or quasistatic) elastic field. Here it is assumed that spatial – temporal variations are slow in comparison with the oscillations in the elastic wave. After substituting the solution of (3.1) into (1.6) and (1.7) and averaging over fast variations, we obtain the corresponding reduced equations for the slowly varying amplitude of the longitudinal-wave second harmonic in the approximation of the given first-harmonic field and of the modulating field:

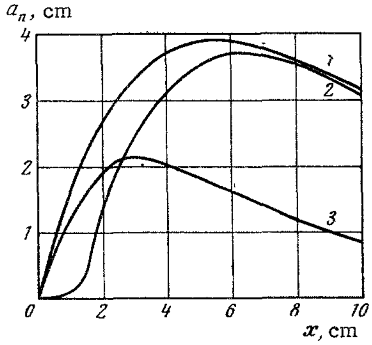

Fig. 1

$$\frac{\partial A_2}{\partial t} + c_l \frac{\partial A_2}{\partial x} = \frac{ikA_2}{\rho_0 c_l}\left(\beta \frac{\partial u_x^m}{\partial x} + \gamma \frac{\partial u_y^m}{\partial y} + \gamma \frac{\partial u_z^m}{\partial z}\right)$$
$$+ \frac{k^2 A_1^2}{4\rho_0 c_l}\left(\beta + 2\tau_1 \frac{\partial u_x^m}{\partial x} + \tau_6 \frac{\partial u_y^m}{\partial y} + \tau_6 \frac{\partial u_z^m}{\partial z}\right) \tag{3.2}$$
$$\beta = 4\mu + 3K + 2A + 6B + 2C, \quad \gamma = K - {}^2/_3\mu + 2B + 2C$$
$$\tau_6 = 24D + 12H + 6G + 6B + 6C$$

Analogously for a transverse wave, polarized along the y axis

$$\frac{\partial A_2}{\partial t} + c_t \frac{\partial A_2}{\partial x} = \frac{ikA_2}{\rho_0 c_t}\left(\beta \frac{\partial u_x^m}{\partial x} + \beta \frac{\partial u_y^m}{\partial y} + \gamma \frac{\partial u_z^m}{\partial z}\right)$$
$$+ \frac{k^2 A_1^2}{4\rho_0 c_t}\left(2\tau_5 \frac{\partial u_y^m}{\partial x} + \tau_7 \left(\frac{\partial u_x^m}{\partial y}\right)\right) \tag{3.3}$$
$$\beta = K + {}^4/_3\mu + A/2 + B, \quad \gamma = K - {}^2/_3\mu + B$$
$$\tau_7 = 6J + 3B + {}^3/_2 A$$

For purposes of simplification in (3.2) and (3.3) dissipative properties of the medium were not taken into account. The first terms on the right sides describe sound modulation by the low-frequency field through the quadratic term [8], and the second terms describe second-harmonic generation in the presence of the low-frequency field. Let us consider some specific forms of the modulating fields.

Let u^m be a uniform tensile-stress field of a rod of strength P_0. The angle between P_0 and the direction of propagation of the wave is θ. Assuming P_0 to lie in the xy plane and taking into account the relations [9]

$$\frac{\partial u_x^m}{\partial x} = \frac{P_0}{E}(\cos^2\theta - \sigma\sin^2\theta), \quad \frac{\partial u_y^m}{\partial y} = \frac{P_0}{E}(\sin^2\theta - \sigma\cos^2\theta)$$
$$\frac{\partial u_z^m}{\partial z} = -\frac{P_0\delta}{E}, \quad \frac{\partial u_x^m}{\partial y} = \frac{\partial u_y^m}{\partial x} = -\frac{P_0(1+\sigma)}{E}\cos\theta\sin\theta \tag{3.4}$$

where E is Young's modulus and σ is the Poisson ratio, we obtain for the real amplitude and phase of the second harmonic of an elastic wave, traveling along the x axis,

$$\varphi_2 = 2\varphi_{10} + m_2 x, \quad a_2 = m_1 a_{10}^2 x \sin m_2 x \,/\, m_2 x \tag{3.5}$$

where m_1 and m_2 are defined by the expressions:

for a longitudinal wave

$$m_1 = \frac{k^2}{4\rho_0 c_l^2}\left\{\beta + \frac{P_0}{E}[2(\tau_1 - \sigma\tau_6)\cos^2\theta + (\tau_6 - \sigma\tau_6 - 2\sigma\tau_1)\sin^2\theta]\right\}$$
$$m_2 = \frac{kP_0}{2\rho_0 c_l^2 E}[(\beta - 2\sigma\gamma)\cos^2\theta + (\gamma - \sigma\gamma - \sigma\beta)\sin^2\theta] \tag{3.6}$$

for a transverse wave

$$m_1 = -\frac{k^2 P_0(1+\sigma)}{4\rho_0 E c_t^2}(2\tau_5 + \tau_7)\sin\theta\cos\theta$$
$$m_2 = \frac{kP_0}{2\rho_0 E c_t^2}(\beta - \sigma\beta - \sigma\gamma) \tag{3.7}$$

From (3.5)-(3.7) we see that the amplitude of the second harmonic depends linearly on the magnitude of the tensile stress P_0 and has an angular dependence determined by the nonlinear properties of the medium.

If u^m (x, y, z, t) is a nonperiodic function of the time and the coordinates, then there occurs phase and amplitude modulation of the second harmonic with the frequency of the external field. The depth of the modulation is determined by the nonlinear elastic properties and by the magnitude of the modulating field. The quadratic nonlinearity of the medium is responsible for the phase modulation [8], and the cubic term is responsible for the amplitude modulation. In its pure form the amplitude modulation due to the cubic nonlinearity can be obtained for second-harmonic generation of a transverse wave by giving the modulating field in the form

$$u_x^m = -u_0^m \sin\theta\cos(\Omega t - \varkappa x\cos\theta - \varkappa y\sin\theta)$$
$$u_y^m = u_0^m \cos\theta\cos(\Omega t - \varkappa x\cos\theta - \varkappa y\sin\theta) \tag{3.8}$$
$$u_z^m = 0$$

The solution to (3.3) is then

$$\varphi_2 = 2\varphi_{10},\ a_2 = 2B\cos(\Omega t - \psi)$$
$$\psi = {}^1/_2 \varkappa x (1 + \cos\theta) + \varkappa y \sin\theta \tag{3.9}$$
$$B = (8\rho_0 c_t^2)^{-1} k^2 a_{10}^2 \varkappa u_0^m (2\tau_5 \cos^2\theta - \tau_7 \sin^2\theta)\, x \sin[{}^1/_2 \varkappa x (1 - \cos\theta)]\, [{}^1/_2 \varkappa x (1 - \cos\theta)]^{-1}$$

The full oscillation for double the frequency can be given in the form

$$U_2 = 2B\cos(\Omega t - \psi)\cos(2\omega t - 2kx + 2\varphi_{10}) = B\cos\times$$
$$\times[(2\omega + \Omega)t - 2kx + 2\varphi_{10} - \psi] + B\cos[(2\omega - \Omega)t - 2kx + 2\varphi_{10} + \psi] \tag{3.10}$$

In distinction to a longitudinal wave here the spectral component with frequency 2ω is missing. This phenomenon can be thought of as an anomalous splitting of an acoustic wave in a periodically inhomogeneous medium.

Thus, going to the nine-constant theory of elasticity, i.e., taking into account terms of fourth order in $\partial u_i/\partial x_k$ in the elastic energy, enables us to estimate the accuracy of the five-constant theory of elasticity when considering third-approximation effects in acoustics. For example, the contribution from the cubic nonlinearity of a solid medium to third-harmonic generation for actually attainable amplitudes of elastic waves and path length of the interaction (allowing for damping) is several percent. In the third approximation we have considered qualitatively these new effects: third shear-wave harmonic generation and second-harmonic generation in the presence of low-frequency fields. In these effects the role of the fourth-order moduli is decisive. This last claim follows from the results of appropriate experiments.

Thus, for example, the magnitudes of the moduli for polycrystalline aluminum, evaluated according to equations (3.5) and (3.6) and the experimental results of Konyukhov [3], give an effective parameter for the quadratic nonlinearity of $\beta/\rho_0 c_l^2 \approx 15$, and for the cubic nonlinearity, of $\tau_1/\rho_0 c_l^2 \approx 1000$. In addition, the four independent elastic moduli of fourth order can be measured by the method whose theory was set forth in [10]. The elastic moduli of third order can be measured by any of the methods set forth in [1, 8, 11, 12].

The phenomena considered here can be used to measure the amplitudes and spectral composition of internal elastic stresses in solids during dynamical tests (of static stress or fatigue failure).

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